

# MATRICES RELATED TO DIRICHLET SERIES

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ABSTRACT. We attach a certain  $n \times n$  matrix  $A_n$  to the Dirichlet series  $L(s) = \sum_{k=1}^{\infty} a_k k^{-s}$ . We study the determinant, characteristic polynomial, eigenvalues, and eigenvectors of these matrices. The determinant of  $A_n$  can be understood as a weighted sum of the first  $n$  coefficients of the Dirichlet series  $L(s)^{-1}$ . We give an interpretation of the partial sum of a Dirichlet series as a product of eigenvalues. In a special case, the determinant of  $A_n$  is the sum of the Möbius function. We disprove a conjecture of Barrett and Jarvis regarding the eigenvalues of  $A_n$ .

## 1. INTRODUCTION

To the Dirichlet series

$$L(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s},$$

we attach the  $n \times n$  matrix

$$D_n = \sum_{k=1}^{\infty} a_k E_n(k),$$

where  $E_n(k)$  is the  $n \times n$  matrix whose  $ij$ th entry is 1 if  $j = ki$  and 0 otherwise.

For example,

$$D_6 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ & a_1 & & a_2 & & a_3 \\ & & a_1 & & & a_2 \\ & & & a_1 & & \\ & & & & a_1 & \\ & & & & & a_1 \end{pmatrix}.$$

Since

$$E_n(k_1)E_n(k_2) = E_n(k_1 k_2)$$

for every  $k_1, k_2 \in \mathbb{N}$ , formally manipulating linear combinations of  $E_n(k)$  is very similar to formally manipulating Dirichlet series. However, because  $E_n(k)$

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is the zero matrix whenever  $k > n$ , the sum defining  $D_n$  is guaranteed to converge. Of course, the  $n \times n$  matrix contains less information than the Dirichlet series. Letting  $n$  tend to infinity produces semi-infinite matrices, the formal manipulation of which is exactly equivalent to formally manipulating Dirichlet series.

Let  $W_n$  be the matrix whose first column is the weight vector  $(0, w_2, w_3, \dots, w_n)^T$  and whose other entries are zeros. Define the  $n \times n$  matrix  $A_n$  (and the special cases  $B_n$  and  $C_n$ ) by

$$\begin{aligned} A_n &= W_n + D_n, \\ (1) \quad B_n &= W_n + D_n \quad \text{when } a_k = 1 \text{ for all } k, \\ C_n &= W_n + D_n \quad \text{when } a_k = 1 \text{ and } w_k = 1 \text{ for all } k. \end{aligned}$$

For example,  $A_6$ ,  $B_6$ , and  $C_6$  are the following three matrices:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ w_2 & a_1 & & a_2 & & a_3 \\ w_3 & & a_1 & & & a_2 \\ w_4 & & & a_1 & & \\ w_5 & & & & a_1 & \\ w_6 & & & & & a_1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ w_2 & 1 & & 1 & & 1 \\ w_3 & & 1 & & & 1 \\ w_4 & & & 1 & & \\ w_5 & & & & 1 & \\ w_6 & & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & & 1 & & 1 \\ 1 & & 1 & & & 1 \\ 1 & & & 1 & & \\ 1 & & & & 1 & \\ 1 & & & & & 1 \end{pmatrix}.$$

We will always assume that  $a_1 = 1$  since this ensures that the Dirichlet series  $\sum a_k k^{-s}$  has a formal inverse and since this is true for many Dirichlet series that arise in number theory. For notational convenience, we set  $w_1 = 1$ , and occasionally we will write  $a(i)$  instead of  $a_i$ . Several authors have studied the matrices  $B_n$  and  $C_n$ . In [4], it was observed that that

$$(2) \quad \det B_n = \sum_{k=1}^n w_k \mu(k),$$

where  $\mu$  is the Möbius  $\mu$ -function. This is a special case of the slightly more general fact (see Theorem 2.1 below) that

$$(3) \quad \det A_n = \sum_{k=1}^n w_k b_k,$$

where the numbers  $b_k$  are the coefficients of the formal series

$$L(s)^{-1} = \sum_{k=1}^{\infty} \frac{b_k}{k^s}.$$

Thus,  $\det A_n$  is a weighted sum of the coefficients of  $L(s)^{-1}$ .

To obtain (2) from (3), choose the Dirichlet series to be the Riemann zeta function  $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$  so that  $a_k = 1$  for all  $k$ . This corresponds to the case of the matrix  $B_n$ . Since  $\zeta(s)^{-1} = \sum_{k=1}^{\infty} \mu(k)k^{-s}$ , where  $\mu$  is the Möbius  $\mu$ -function, it follows that  $b_k = \mu(k)$ . One particularly intriguing choice for  $w_k$  is  $w_k = k^{-s}$ . Then (3) results in the truncated Dirichlet series

$$\det A_n = \sum_{k=1}^n \frac{b_k}{k^s}.$$

As the asymptotic growth of sums of the type in equation (3) is important to analytic number theory, representing those sums in terms of determinants becomes very interesting.

Recall that the Riemann hypothesis is equivalent to the statement

$$\sum_{k=1}^n \mu(k) = O(n^{1/2+\epsilon}),$$

for every positive  $\epsilon$ . Thus, the Riemann hypothesis is equivalent to

$$\det C_n = O(n^{1/2+\epsilon}),$$

for every positive  $\epsilon$ .

In [1], Barrett, Forcade, and Pollington expressed the characteristic polynomial of  $C_n$  as

$$(4) \quad p_n(x) = (x-1)^{n-r-1} \left( (x-1)^{r+1} - \sum_{k=1}^r v(n, k)(x-1)^{r-k} \right),$$

where  $r = \lfloor \log_2 n \rfloor$  and where the coefficients  $v(n, k)$  were described in terms of directed graphs. We will refer to the eigenvalue 1, whose multiplicity is  $n - r - 1$ , as the *trivial* eigenvalue. The eigenvalues  $\lambda \neq 1$  will be called *nontrivial* eigenvalues. In Theorem 3.2 we extend this result by determining

the characteristic polynomial of the more general matrix  $A_n$ . In [1], it was shown that the spectral radius  $\rho(C_n)$  of  $C_n$  is asymptotic to  $\sqrt{n}$ .

Barrett and Robinson [5] determined that the sizes of the Jordan blocks of  $B_n$  corresponding to the trivial eigenvalue 1 are

$$[\log_2(n/3)] + 1, [\log_2(n/5)] + 1, \dots, [\log_2(n/\{n\})] + 1,$$

where  $\{n\}$  denotes the greatest odd integer  $\leq n$ . Theorem 4.1 of this paper shows that each nontrivial eigenvalue of  $A_n$  is simple and expresses a basis for the one-dimensional eigenspace in terms of a recursion involving the coefficients of  $p_m(x)$  for  $m < n$ , enhancing our understanding of the Jordan form of  $A_n$ . Theorem 4.2 gives a similar result for the transpose of  $A_n$ .

The coefficients of the characteristic polynomial of  $C_n$  are related to the Riemann zeta function as follows: If  $(\zeta(s) - 1)^k$  is expressed as a Dirichlet series  $\sum_{m=1}^{\infty} \frac{c(m,k)}{m^s}$  so that

$$\frac{1}{1 + (\zeta(s) - 1)} = \sum_{k=0}^{\infty} (-1)^k (\zeta(s) - 1)^k = \sum_{k=0}^{\infty} (-1)^k \left( \sum_{m=1}^{\infty} \frac{c(m,k)}{m^s} \right),$$

then

$$v(n, k) = \sum_{j \leq n} c(j, k).$$

Evaluating  $p_n(x)$  at  $x = 0$  gives the fundamental relationship

$$\det C_n = \sum_{i=1}^n \mu(i) = \prod_{\lambda \text{ nontrivial}} \lambda = \sum_{k=0}^{[\log_2 n]} (-1)^k v(n, k),$$

where  $v(n, 0)$  is defined to equal 1.

Barrett and Jarvis [2] showed that  $C_n$  has two large real eigenvalues  $\lambda_{\pm}$  satisfying

$$(5) \quad \lambda_{\pm} = \pm \sqrt{n} + \log \sqrt{n} + \gamma - 1/2 + O\left(\frac{\log^2 n}{\sqrt{n}}\right),$$

and that the remaining  $[\log_2 n] - 1$  small nontrivial eigenvalues satisfy

$$|\lambda| < \log_{2-\epsilon} n$$

for any small positive  $\epsilon$  and sufficiently large  $n$ . Based on numerical evidence for various values of  $n$  as large as  $n = 10^6$ , they also made the following two-part conjecture:

**Conjecture 1.1** (Barrett and Jarvis [2]). *The small nontrivial eigenvalues  $\lambda$  of  $C_n$  satisfy*

- (i)  $|\lambda| < 1$ , and
- (ii)  $\operatorname{Re}(\lambda) < 1$ .

The statement  $\operatorname{Re}(\lambda) < 1$  is, of course, weaker than the statement  $|\lambda| < 1$ .

Vaughan [6] refined the asymptotic formula (5) for the two large eigenvalues and showed, unconditionally, that the small eigenvalues satisfy

$$|\lambda| \ll (\log n)^{2/5},$$

and, upon the Riemann hypothesis, that the small eigenvalues satisfy

$$(6) \quad |\lambda| \ll \log \log(2 + n).$$

He later showed [7] that  $C_n$  has nontrivial eigenvalues arbitrarily close to 1 for sufficiently large  $n$ , suggesting that a proof of Conjecture 1.1 would likely be quite subtle.

Investigations of the Redheffer matrix have been extended to group theory by Humphries [3] and to partially ordered sets by Wilf [8].

In Theorem 5.3, we resolve Conjecture 1.1 by showing that both parts are false. There exist values of  $n$  for which a small eigenvalue  $\lambda$  satisfies both  $|\lambda| > 1$  and  $\operatorname{Re}(\lambda) > 1$ . To accomplish this we computed the characteristic polynomials for  $A_n$  for values of  $n$  as large as  $n = 2^{36}$ , which we describe in §5.

## 2. THE DETERMINANT OF $A_n$

We now find the determinant of  $A_n$ .

**Theorem 2.1.** *Let  $D_n$  be the Dirichlet matrix associated with the formal Dirichlet series  $L(s) = \sum_{k=1}^{\infty} a_k k^{-s}$  where  $a_1 = 1$ , and write  $L(s)^{-1} = \sum_{k=1}^{\infty} b_k k^{-s}$ . Let  $W_n$  be the matrix whose first column is  $(0, w_2, \dots, w_n)^T$  and whose other*

entries are zero. Let  $A_n = W_n + D_n$  as in (1). Also, let  $\tilde{A}_n = W_n + D_n^{-1}$ . Then

$$(7) \quad \det(A_n) = \sum_{k=1}^n w_k b_k \quad \text{and} \quad \det(\tilde{A}_n) = \sum_{k=1}^n w_k a_k.$$

**Corollary 2.2.** *The choice  $w_k = 1$  produces partial sums of coefficients of Dirichlet series:*

$$(8) \quad \det A_n = \sum_{k=1}^n b_k \quad \text{and} \quad \det \tilde{A}_n = \sum_{k=1}^n a_k.$$

**Corollary 2.3.** *The choice  $w_k = k^{-s}$  gives truncations of the Dirichlet series  $L(s)^{-1}$  and  $L(s)$ :*

$$(9) \quad \det A_n = \sum_{k=1}^n \frac{b_k}{k^s} \quad \text{and} \quad \det \tilde{A}_n = \sum_{k=1}^n \frac{a_k}{k^s}.$$

If  $s$  is a complex number at which  $L(s)$  and  $L(s)^{-1}$  converge,

$$\lim_{n \rightarrow \infty} \det A_n = L(s)^{-1} \quad \text{and} \quad \lim_{n \rightarrow \infty} \det \tilde{A}_n = L(s).$$

So, Corollary 2.3 says that we may interpret  $\det A_n$  and  $\det \tilde{A}_n$  as approximating values of Dirichlet series. Since the determinant is the product of the eigenvalues, this relates values of Dirichlet series with eigenvalues of matrices.

*Proof of Theorem 2.1.* This is essentially the same argument as the one given in Redheffer's note [4] where he found the determinant of  $B_n$ . Since  $D_n$  is upper triangular with diagonal entry 1,  $\det D_n = \det D_n^{-1} = 1$ . Then

$$\det A_n = \det D_n^{-1} \det A_n = \det D_n^{-1} \det(W_n + D_n) = \det(D_n^{-1}W_n + I_n).$$

The matrix  $D_n^{-1}W_n$  has zeros in columns 2 through  $n$ , and its  $(1,1)$ -entry is  $\sum_{k=2}^n w_k b_k$ . Thus,  $\det A_n$  equals the  $(1,1)$  entry of  $D_n^{-1}W_n + I_n$  which is  $\sum_{k=1}^n w_k b_k$ . Replacing  $D_n$  with  $D_n^{-1}$  in the argument gives  $\det \tilde{A}_n = \sum_{k=1}^n w_k a_k$ .  $\square$

3. THE CHARACTERISTIC POLYNOMIAL OF  $A_n$ 

The characteristic polynomial  $p_n(x) = \det(I_n x - A_n)$  plays a significant role. Previously,  $p_n(x)$  was obtained for the special case  $C_n$  in [1] and [6]. In this section, we will determine the characteristic polynomial of the more general matrix  $A_n$ . The following definition will be instrumental in describing both the characteristic polynomial of  $A_n$  and its eigenvectors.

**Definition.** For integers  $n \geq 1$  and  $k \geq 0$ , we define  $d(n, k)$  to be the Dirichlet series coefficients of  $(L(s) - 1)^k$ . That is,

$$(10) \quad (L(s) - 1)^k = \left( \sum_{n=2}^{\infty} \frac{a_n}{n^s} \right)^k = \sum_{n=1}^{\infty} \frac{d(n, k)}{n^s}.$$

We define  $v(n, k)$  and  $v_\ell(n, k)$  to be the weighted sums:

$$(11) \quad \begin{aligned} v(n, k) &= \sum_{j \leq n} w(j) d(j, k), \quad \text{and} \\ v_\ell(n, k) &= \sum_{j \leq n} w(j\ell) d(j, k). \end{aligned}$$

Several cases of this definition are important to keep in mind:  $d(1, 0) = 1$  and  $d(n, 0) = 0$  for  $n > 1$ ; also, both  $d(n, k)$  and  $v(n, k)$  are zero if  $n < 2^k$  since a number smaller than  $2^k$  cannot be written as a product of  $k$  nontrivial factors.

From the definition of  $d(n, k)$ ,

$$\sum_{n=1}^{\infty} \frac{d(n, k)}{n^s} = \left( \sum_{n=2}^{\infty} \frac{a_n}{n^s} \right) \left( \sum_{n=2}^{\infty} \frac{a_n}{n^s} \right)^{k-1} = \left( \sum_{n=2}^{\infty} \frac{a_n}{n^s} \right) \left( \sum_{n=1}^{\infty} \frac{d(n, k-1)}{n^s} \right),$$

which immediately gives the elementary recurrence relation:

**Lemma 3.1.** *If  $k \geq 1$ , then*

$$(12) \quad d(n, k) = \sum_{\substack{i|n \\ 1 < i}} a(i) d(n/i, k-1) = \sum_{\substack{j|n \\ j < n}} a(n/j) d(j, k-1).$$

**Theorem 3.2.** *The characteristic polynomial  $p_n(x) = \det(xI_n - A_n)$  is*

$$(13) \quad p_n(x) = (x-1)^{n-r-1} \left( (x-1)^{r+1} - \sum_{k=1}^r v(n, k) (x-1)^{r-k} \right),$$

where  $r = \lfloor \log_2 n \rfloor$ . Consequently, if  $v(n, r) \neq 0$ , the algebraic multiplicity of the trivial eigenvalue  $\lambda = 1$  is  $n - r - 1$ .

*Proof.* We will use the cofactor expansion to calculate the characteristic polynomial  $p_n(x) = \det(xI_n - A_n)$ . Write  $M_n = xI_n - A_n$  and let

$$M_n(i_1, \dots, i_s \mid j_1, \dots, j_t)$$

denoted the matrix obtained by removing the rows indexed by  $i_1, \dots, i_s$  and the columns indexed by  $j_1, \dots, j_t$  from  $M_n$ . The cofactor expansion of the determinant along the first column is

$$p_n(x) = (x-1)^n + \sum_{k=2}^n (-1)^k w_k \det M_n(k \mid 1).$$

The matrix  $M_n(k \mid 1)$  is a block matrix whose upper left  $(k-1) \times (k-1)$  block is  $M_k(k \mid 1)$ , whose lower left  $(n-k) \times (k-1)$  block is zero, and whose lower right  $(n-k) \times (n-k)$  block is upper triangular with diagonal entries  $x-1$ . Thus

$$\det M_n(k \mid 1) = (x-1)^{n-k} \det M_k(k \mid 1),$$

where we understand  $\det M_1(1 \mid 1)$  to be 1, and

$$(14) \quad p_n(x) = (x-1)^n + \sum_{k=2}^n (-1)^k w_k (x-1)^{n-k} \det M_k(k \mid 1).$$

The  $\ell$ th entry in the last column of  $M_k(k \mid 1)$  is  $-a_{n/\ell}$  if  $\ell$  divides  $k$ ; otherwise, it is zero. Then the cofactor expansion of  $\det M_k(k \mid 1)$  along the last column is

$$\det M_k(k \mid 1) = \sum_{\substack{\ell \mid k \\ \ell < k}} (-1)^{k+\ell} a_{k/\ell} \det M_k(\ell, k \mid 1, k).$$

The matrix  $M_k(\ell, k \mid 1, k)$  is also a block matrix. Since the upper left  $(\ell-1) \times (\ell-1)$  block is  $M_\ell(\ell \mid 1)$ , the lower left  $(k-\ell-1) \times (\ell-1)$  block is zero, and the lower right  $(k-\ell-1) \times (k-\ell-1)$  block is upper triangular with diagonal entries  $x-1$ ,

$$\det M_k(\ell, k \mid 1, k) = (x-1)^{k-\ell-1} \det M_\ell(\ell \mid 1).$$



This shows that

$$(15) \quad \det M_k(k \mid 1) = \sum_{\substack{\ell \mid k \\ \ell < k}} (-1)^{k+\ell} a_{k/\ell} (x-1)^{k-\ell-1} \det M_\ell(\ell \mid 1).$$

In other words, the quantity  $q_k(x) = (-1)^{k-1} M_k(k \mid 1)$  satisfies the recurrence relation:

$$(16) \quad \begin{aligned} q_1(x) &= 1, \\ q_k(x) &= \sum_{\substack{\ell \mid k \\ \ell < k}} a_{k/\ell} (x-1)^{k-\ell-1} q_\ell(x) \quad \text{for } k > 1. \end{aligned}$$

On the other hand, consider the polynomial  $t_\ell(x)$  defined by

$$(17) \quad t_\ell(x) = \sum_{j \geq 0} d(\ell, j) (x-1)^{\ell-j-1}.$$

Then  $t_1(x) = 1$ . For  $\ell > 1$ , the term in the sum corresponding to  $j = 0$  is zero since  $d(\ell, 0) = 0$  in that case. For  $k > 1$ , calculating the right hand side of (16) with  $t_\ell(x)$  in place of  $q_\ell(x)$  and applying Lemma 3.1 gives

$$\begin{aligned} \sum_{\substack{\ell \mid k \\ \ell < k}} a_{k/\ell} (x-1)^{k-\ell-1} t_\ell(x) &= \sum_{\substack{\ell \mid k \\ \ell < k}} \sum_{j \geq 0} a_{k/\ell} d(\ell, j) (x-1)^{k-j-2} \\ &= \sum_{j \geq 0} d(k, j+1) (x-1)^{k-j-2} \\ &= \sum_{j \geq 1} d(k, j) (x-1)^{k-j-1} \\ &= t_k(x). \end{aligned}$$

Since  $t_k(x)$  and  $q_k(x)$  both satisfy the same recurrence relations, they are equal.

This shows that

$$(-1)^{k-1} M_k(k \mid 1) = q_k(x) = t_k(x) = \sum_{j \geq 0} d(k, j) (x-1)^{k-j-1}.$$

Substituting the last expression into (14) gives

$$p_n(x) = (x-1)^n - \sum_{k=2}^n \sum_{j \geq 1} w_k d(k, j) (x-1)^{n-j-1}$$

$$= (x-1)^n - \sum_{j \geq 1} v(n, j)(x-1)^{n-j-1}.$$

Since  $v(n, j) = 0$  for  $j > r = \lfloor \log_2(n) \rfloor$ , this is

$$\begin{aligned} p_n(x) &= (x-1)^n - \sum_{j=1}^r v(n, j)(x-1)^{n-j-1} \\ &= (x-1)^{n-r-1} \left( (x-1)^{r+1} - \sum_{j=1}^r v(n, j)(x-1)^{r-j} \right), \end{aligned}$$

which proves the theorem.  $\square$

#### 4. THE EIGENVECTORS OF $A_n$

**Theorem 4.1.** *Let  $\lambda \neq 1$  be a nontrivial eigenvalue of  $A_n$ . Then  $\lambda$  is a simple eigenvalue, and a basis for the one dimensional eigenspace of  $A_n$  associated with  $\lambda$  is the vector*

$$u = [\lambda - 1, X_2(\lfloor n/2 \rfloor), X_3(\lfloor n/3 \rfloor), X_4(\lfloor n/4 \rfloor), \dots, X_n(\lfloor n/n \rfloor)]^T$$

where

$$X_j(q) = \sum_{k \geq 0} \frac{v_j(q, k)}{(\lambda - 1)^k} = 1 + \frac{v_j(q, 1)}{\lambda - 1} + \frac{v_j(q, 2)}{(\lambda - 1)^2} + \frac{v_j(q, 3)}{(\lambda - 1)^3} + \dots$$

*Proof of Theorem 4.1.* For  $i \geq 2$ , the  $i$ th entry of  $A_n u$  is

$$\begin{aligned} (A_n u)_i &= w_i(\lambda - 1) + \sum_{1 \leq \ell \leq n/i} a_\ell u_{\ell i} \\ &= w_i(\lambda - 1) + \sum_{1 \leq \ell \leq n/i} a_\ell X_{\ell i}(\lfloor n/(\ell i) \rfloor) \\ &= w_i(\lambda - 1) + \sum_{1 \leq \ell \leq n/i} a_\ell \sum_{\substack{k \geq 0 \\ 1 \leq m \leq n/i}} w(i\ell m) d(m, k) (\lambda - 1)^{-k} \\ &= w_i(\lambda - 1) + \sum_{k \geq 0} \left( \sum_{\substack{1 \leq \ell \leq n/i \\ 1 \leq m \leq n/(\ell i)}} a(\ell) w(i\ell m) d(m, k) \right) (\lambda - 1)^{-k} \\ &= w_i(\lambda - 1) + \sum_{k \geq 0} \left( \sum_{1 \leq t \leq n/i} w(it) \sum_{s|t} a(t/s) d(s, k) \right) (\lambda - 1)^{-k} \quad [\text{set } t = i\ell] \\ &= w_i(\lambda - 1) + \sum_{k \geq 0} \sum_{1 \leq t \leq n/i} w(it) (d(t, k) + d(t, k+1)) (\lambda - 1)^{-k} \quad [\text{by (12)}] \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \geq 0} v_i(\lfloor n/i \rfloor, k)(\lambda - 1)^{-k} + w_i(\lambda - 1) + \sum_{k \geq 1} v_i(\lfloor n/i \rfloor, k)(\lambda - 1)^{-k+1} \\
&= \sum_{k \geq 0} v_i(\lfloor n/i \rfloor, k)(\lambda - 1)^{-k} + (\lambda - 1) \sum_{k \geq 0} v_i(\lfloor n/i \rfloor, k)(\lambda - 1)^{-k} \\
&= \lambda \sum_{k \geq 0} v_i(\lfloor n/i \rfloor, k)(\lambda - 1)^{-k} \\
&= \lambda X_i(\lfloor n/i \rfloor) \\
&= \lambda u_i.
\end{aligned}$$

In the calculation for  $(A_n u)_i$  with  $i \geq 2$ , the term  $a_i u_{\ell i}$  when  $\ell = 1$  was equal to  $a_i X_i(\lfloor n/i \rfloor)$ , but this term should be omitted from the case  $i = 1$ . Taking this into account and going to the second to last step of the previous calculation gives

$$\begin{aligned}
(A_n u)_1 &= \lambda X_1(n) - X_1(n) \\
&= (\lambda - 1) \left( 1 + \sum_{k \geq 1} v(n, k)(\lambda - 1)^{-k} \right) \\
&= (\lambda - 1) [1 + (\lambda - 1)] \quad \text{by Theorem 3.2} \\
&= \lambda(\lambda - 1) \\
&= \lambda u_1.
\end{aligned}$$

This shows that the vector  $u$  is a nonzero eigenvector for  $\lambda$ .

To see why the eigenspace of  $\lambda$  is one-dimensional, consider the submatrix of  $A_n - \lambda I$  obtained by deleting the first row and column. This  $(n-1) \times (n-1)$  matrix is upper triangular with nonzero entries on the diagonal. Hence, it is invertible implying that the rank of  $A_n - \lambda I_n$  is  $\geq n-1$ . Since we found a nontrivial eigenvector, the nullity is  $\geq 1$ . So, the nullity of  $A_n - \lambda I$  must be exactly one. This completes the proof.  $\square$

**Theorem 4.2.** *Let  $\lambda \neq 1$  be a nontrivial eigenvalue of  $A_n$ . A basis for the one dimensional eigenspace of  $A_n^T$  associated with  $\lambda$  is the vector*

$$(18) \quad v = [1, Y_\lambda(2), Y_\lambda(3), \dots, Y_\lambda(n)]^T.$$

where

$$Y_\lambda(q) = \sum_{k \geq 0} \frac{d(q, k)}{(\lambda - 1)^k} = d(q, 0) + \frac{d(q, 1)}{\lambda - 1} + \frac{d(q, 2)}{(\lambda - 1)^2} + \cdots.$$

Interestingly, the algebraic expression for  $v$  does not explicitly rely on the symbols  $w_2, \dots, w_n$  in the first column of  $A_n$ . However, altering  $w_2, \dots, w_n$  changes the possible numeric values of  $\lambda$ .

*Proof of Theorem 4.2.* For  $i \geq 2$ , the  $i$ th entry of  $A_n^T v$  is

$$\begin{aligned} (A_n^T v)_i &= \sum_{\ell | i} a(i/\ell) Y(\ell) \\ &= \sum_{k \geq 0} \sum_{\ell | i} a(i/\ell) d(\ell, k) (\lambda - 1)^{-k} \\ &= \sum_{k \geq 0} [d(i, k) + d(i, k+1)] (\lambda - 1)^{-k} \quad \text{by (12)} \\ &= Y(i) + (\lambda - 1) \sum_{k \geq 1} d(i, k) (\lambda - 1)^{-k} \\ &= Y(i) + (\lambda - 1) Y(i) \quad [\text{since } d(i, 0) = 0] \\ &= \lambda Y(i). \end{aligned}$$

The first entry of  $A_n^T v$  is

$$\begin{aligned} (A_n^T)_i &= \sum_{1 \leq j \leq n} w_j Y(j) \\ &= \sum_{k \geq 0} \sum_{1 \leq j \leq n} w_j d(j, k) (\lambda - 1)^{-k} \\ &= \sum_{k \geq 0} v(n, k) (\lambda - 1)^{-k} \\ &= 1 + \sum_{k \geq 1} v(n, k) (\lambda - 1)^{-k} \\ &= 1 + (\lambda - 1) \quad [\text{by Theorem 3.2}] \\ &= \lambda v_1. \end{aligned}$$

This shows that  $v = [Y(1), \dots, Y(n)]^T$  is a nonzero eigenvector of  $A_n^T$ . The dimension of the eigenspace is one, as explained in the proof of Theorem 4.1.  $\square$

## 5. COMPUTING EIGENVALUES OF $C_n$ FOR LARGE $n$

Theorem 3.2 expresses the characteristic polynomial of the matrix  $A_n$  in terms of the numbers  $v(n, k)$ . In this section, we will explain how to explicitly calculate the characteristic polynomial  $p_n(x)$  for large values of  $n$  for the special case  $C_n$  in which  $w_i = a_i = 1$  for all  $i$ . The method given below in Theorem 5.2 was used to find  $p_n(x)$  for  $n$  as large as  $n = 2^{36}$  in a few hours on a desktop computer. To accomplish this, it is necessary to use a more efficient algorithm for finding the coefficients than a brute force approach based directly on the definition of matrix  $C_n$ . Even with Theorem 3.2 we need a better method for computing  $v(n, k)$  than the direct application of the definition of  $v(n, k)$  in (11).

**Lemma 5.1.** *Suppose  $a_\ell = w_\ell = 1$  for all  $\ell$ . If  $1 \leq 2^k \leq n$ , then*

$$(19) \quad v(n, k) = \sum_{i>1} v(\lfloor \frac{n}{i} \rfloor, k-1) = \sum_{j<n} (\lfloor \frac{n}{j} \rfloor - \lfloor \frac{n}{j+1} \rfloor) v(j, k-1).$$

If both  $a_k = 1$  and  $w_k = 1$  for all  $k$ , then  $v(n, k)$  represents the number of ways to form products of  $k$  nontrivial factors whose product is  $\leq n$  and where order matters. In this case,  $v(n, k)$  represents a count of lattices points in  $k$ -dimensional space:

$$(20) \quad v(n, k) = |\{(\ell_1, \dots, \ell_k) \in \mathbb{Z}^k : \ell_1 \ell_2 \cdots \ell_k \leq n \text{ and } \ell_i \geq 2 \text{ for all } i\}|.$$

*Proof.* The first equality in (19) is evident from (20) by letting one component of  $(\ell_1, \dots, \ell_k)$ , say  $\ell_k$ , be the index of summation  $i$ . The second equality in (19) is obtained by re-indexing the sum over the distinct values of  $j = \lfloor n/i \rfloor$ . For a given positive integer  $j$ ,

$$j = \left\lfloor \frac{n}{i} \right\rfloor \quad \Leftrightarrow \quad j \leq \frac{n}{i} < j+1 \quad \Leftrightarrow \quad \frac{n}{j+1} < i \leq \frac{n}{j}.$$

Thus, the number of distinct  $i$  for which  $\lfloor n/i \rfloor = j$  is  $\lfloor \frac{n}{j} \rfloor - \lfloor \frac{n}{j+1} \rfloor$ .  $\square$

The first recursion formula in (19) is computationally inefficient since there can be many distinct values of  $i_1$  and  $i_2$  such that  $\lfloor n/i_1 \rfloor = \lfloor n/i_2 \rfloor$ . The second is inefficient since there can be many values of  $j$  such that  $\lfloor n/j \rfloor - \lfloor n/(j+1) \rfloor$  is zero. The next theorem helps to remove this redundancy by rewriting the summation to have significantly fewer terms.

**Theorem 5.2.** *Assume  $a_\ell = w_\ell = 1$  for all  $\ell$ . Suppose  $1 \leq 2^k \leq n$  and  $k \geq 1$ . Then*

$$(21) \quad v(n, k) = \sum_{i=2}^s v\left(\left\lfloor \frac{n}{i} \right\rfloor, k-1\right) + \sum_{j=2^{k-1}}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n}{j+1} \right\rfloor\right) v(j, k-1),$$

where  $s = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \right\rfloor$ .

*Proof.* This argument applies the hyperbola method from analytic number theory. Rewrite (19) as

$$(22) \quad v(n, k) = \sum_{\lfloor n/i \rfloor \geq \lfloor \sqrt{n} \rfloor + 1} v\left(\left\lfloor \frac{n}{i} \right\rfloor, k-1\right) + \sum_{\lfloor n/i \rfloor \leq \lfloor \sqrt{n} \rfloor} v\left(\left\lfloor \frac{n}{i} \right\rfloor, k-1\right),$$

where the index  $i$  in each summation satisfies  $2 \leq i \leq \lfloor n/2^{k-1} \rfloor$ . In the first summation, since both  $i$  and  $\lfloor \sqrt{n} \rfloor + 1$  are integers,

$$\left\lfloor \frac{n}{i} \right\rfloor \geq \lfloor \sqrt{n} \rfloor + 1 \Leftrightarrow \frac{n}{i} \geq \lfloor \sqrt{n} \rfloor + 1 \Leftrightarrow i \leq \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \Leftrightarrow i \leq \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \right\rfloor.$$

This gives the value  $s = \lfloor n/(\lfloor \sqrt{n} \rfloor + 1) \rfloor$  in the first summation in equation (21).

In the second summation in (22), we re-index the sum over the distinct values of  $j = \lfloor n/i \rfloor \leq \lfloor \sqrt{n} \rfloor$ . For a given positive integer  $j$ ,

$$j = \left\lfloor \frac{n}{i} \right\rfloor \Leftrightarrow j \leq \frac{n}{i} < j+1 \Leftrightarrow \frac{n}{j+1} < i \leq \frac{n}{j}.$$

Thus, the number of distinct  $i$  for which  $\lfloor n/i \rfloor = j$  is  $\lfloor \frac{n}{j} \rfloor - \lfloor \frac{n}{j+1} \rfloor$ , allowing the second summation in (22) to be written as

$$\sum_{j=2^{k-1}}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n}{j+1} \right\rfloor\right) v(j, k-1).$$

This proves (21). □

It is interesting to note that  $s$  in Lemma 5.2 is equal to either  $\lfloor \sqrt{n} \rfloor$  or  $\lfloor \sqrt{n} \rfloor - 1$  according to

$$s = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \right\rfloor = \begin{cases} \lfloor \sqrt{n} \rfloor & \text{if } n - \lfloor \sqrt{n} \rfloor^2 \geq \lfloor \sqrt{n} \rfloor, \\ \lfloor \sqrt{n} \rfloor - 1 & \text{if } n - \lfloor \sqrt{n} \rfloor^2 < \lfloor \sqrt{n} \rfloor. \end{cases}$$

**Theorem 5.3.** *Conjecture 1.1 is false. There exist values of  $n$  for which a small eigenvalue  $\lambda$  of  $C_n$  satisfies both  $|\lambda| > 1$  and  $\operatorname{Re}(\lambda) > 1$ .*

*Proof.* The characteristic polynomial  $p_n(x)$  of the general matrix  $A_n$  was given in Theorem 3.2. By implementing the recursive formula in Theorem 5.2, we were able to calculate the characteristic polynomial for the special case  $C_n$  for relatively large values of  $n$ , such as  $n = 2^{36}$ , within a few hours on a desktop computer.

A table listing the maximum absolute value and real part of small nontrivial eigenvalues of  $C_n$  for  $n = 10^6$  and  $n = 2^r$  with  $28 \leq r \leq 36$  is given below:

$n$	$\max\{ \lambda \}$	$\max\{\operatorname{Re}(\lambda)\}$
$10^6 = 1,000,000$	0.983108	0.983108
$2^{28} = 268,435,456$	0.998885	0.998739
$2^{29} = 536,870,912$	0.999120	0.998989
$2^{30} = 1,073,741,824$	0.999324	0.999206
$2^{31} = 2,147,483,648$	0.999501	0.999395
$2^{32} = 4,294,967,296$	0.999676	0.999560
$2^{33} = 8,589,934,592$	1.002646	0.999704
$2^{34} = 17,179,869,184$	1.005213	0.999829
$2^{35} = 34,359,738,368$	1.007423	0.999939
$2^{36} = 68,719,476,736$	1.031192	1.000036

The example with  $n = 2^{36}$  provides a counter-example to both parts of Conjecture 1.1. A sample of the coefficients  $v(n, k)$  of  $p_n(x)$  for  $n = 10^6$ ,  $n = 2^{28}$ , and  $n = 2^{36}$  is given in Table 1.  $\square$

TABLE 1. Values of  $v(n, k)$  for  $n = 10^6$ ,  $n = 2^{28}$ , and  $n = 2^{36}$ 

$k$	$v(10^6, k)$	$v(2^{28}, k)$	$v(2^{36}, k)$
1	999999	268435455	68719476735
2	11970035	4714411991	1587951104025
3	67120491	39550266080	17712699735807
4	233959922	210866000001	127006997038631
5	567345854	801946179797	657738684402616
6	1015020739	2314766752399	2620541404211325
7	1386286166	5267935378357	8354699452581663
8	1475169888	9693670870002	21888970237054221
9	1237295133	14675212443928	48028484118248949
10	822451796	18500845515388	89496511738284187
11	433656192	19585798031078	143118705146069804
12	180821164	17506983509953	197979547265239162
13	59146673	13254336924806	238336089820847725
14	14935574	8508754910066	250812663743567239
15	2829114	4628591443629	231467885026020936
16	383693	2128656115076	187727209728498411
17	34630	824357770148	133949812310943213
18	1672	267263904116	84103735312636462
19	20	71941723387	46433832280215021
20		15889930335	22505741596654059
21		2830811858	9551600816612963
22		396537923	3536981261202340
23		42162106	1137490727898326
24		3284753	315879734318303
25		177731	75228001661856
26		4707	15244074212812
27		55	2604780031507
28		1	371154513760
29			43388420848
30			4049932603
31			290175811
32			15487073
33			582143
34			9555
35			71
36			1

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